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# A note on the universal $\$sl_2\$$ invariant of Brunnian bottom tangles (Intelligence of Low-dimensional Topology)

AUTHOR(S):

Suzuki, Sakie

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## A note on the universal $sl_2$ invariant of Brunnian bottom tangles

Sakie Suzuki

Research Institute for Mathematical Sciences, Kyoto University

This note is based on the author's talk "On the universal  $sl_2$  invariant of Brunnian bottom tangles" given in the conference "Intelligence of Low-dimensional Topology" held at the RIMS, Kyoto University, during May 16– May 18, 2012. The details appear in [11]

### 1 Introduction

The universal  $sl_2$  invariant has a universality property for the colored Jones polynomial of links [4, 5, 7, 8]. We are interested in the relationship between topological properties of tangles and links and algebraic properties of the universal  $sl_2$  invariant and the colored Jones polynomials.

A *bottom tangle* is a tangle in a cube consisting of only arc components such that each boundary point is on the bottom and the two boundary points of each arc are adjacent to each other, see Figure 1 (a) for example. The closure of a bottom tangle is defined as in Figure 1 (b).

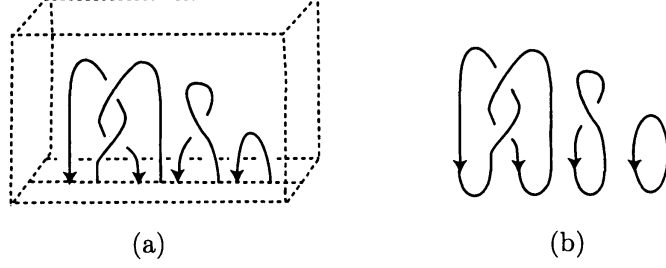
The universal  $sl_2$  invariant of  $n$ -component bottom tangles takes values in the completed  $n$ -fold tensor power  $U_h(sl_2)^{\hat{\otimes} n}$  of the quantized enveloping algebra  $U_h(sl_2)$ . The colored Jones polynomial of a link  $L$  is obtained from the universal  $sl_2$  invariant of a bottom tangle whose closure is  $L$ , by taking the quantum traces associated with the representations attached to the components of links [2].

A bottom tangle is called *ribbon* if its closure is a ribbon link (cf. [3, 9]). A bottom tangle is called *boundary* if its components admit mutually disjoint Seifert surfaces of bottom tangles (cf. [3, 10]). A bottom tangle  $T$  is called *Brunnian* if every proper subtuple of  $T$  is *trivial*, i.e., looks like  $\cap \cdots \cap$ .

Habiro [3] proved that the universal  $sl_2$  invariant of  $n$ -component, algebraically-split, 0-framed bottom tangles takes values in a certain small subalgebra of  $U_h(sl_2)^{\hat{\otimes} n}$ . The present author proved improvements of Habiro's result in the special cases of ribbon bottom tangles [9], boundary bottom tangles [10], and Brunnian bottom tangles [11]. In [9, 10, 11], she also proved that the colored Jones polynomials of ribbon links, boundary links, and Brunnian links take values in certain small ideals of  $\mathbb{Z}[q, q^{-1}]$ .

In this note, we give a survey on the paper [11].

This note is organized as follows. In Section 2, we recall the definitions of  $U_h(sl_2)$  and the universal  $sl_2$  invariant of bottom tangles. In Section 3, we give the main result for the universal  $sl_2$  invariant of Brunnian bottom tangles (Theorem 3.5), and in Section 4, we give an application of Theorem 3.5 to the colored Jones polynomial of Brunnian links (Theorem 4.2).

Figure 1: (a) A bottom tangle  $T$ , (b) The closure link of  $T$ 

## 2 Universal $sl_2$ invariant of bottom tangles

In this section, we recall the definition of  $U_h(sl_2)$  and the universal  $sl_2$  invariant of bottom tangles.

In what follows, we use the following  $q$ -integer notations.

$$\begin{aligned} \{i\}_q &= q^i - 1, & \{i\}_{q,n} &= \{i\}_q \{i-1\}_q \cdots \{i-n+1\}_q, & \{n\}_q! &= \{n\}_{q,n}, \\ [i]_q &= \{i\}_q / \{1\}_q, & [n]_q! &= [n]_q [n-1]_q \cdots [1]_q, & \left[ \begin{matrix} i \\ n \end{matrix} \right]_q &= \{i\}_{q,n} / \{n\}_q!, \end{aligned}$$

for  $i \in \mathbb{Z}, n \geq 0$ .

### 2.1 Quantized enveloping algebra $U_h(sl_2)$ and universal $R$ matrix

We recall the definition of the universal enveloping algebra  $U_h(sl_2)$ .

We denote by  $U_h = U_h(sl_2)$  the  $h$ -adically complete  $\mathbb{Q}[[h]]$ -algebra, topologically generated by  $H, E$ , and  $F$ , defined by the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.$$

Set

$$\begin{aligned} \tilde{E}^{(n)} &= (q^{-1/2} E)^n / [n]_q!, & \tilde{F}^{(n)} &= F^n K^n / [n]_q! \in U_h, \\ e &= (q^{1/2} - q^{-1/2}) E, & f &= (q - 1) F K \in U_h, \\ D &= q^{\frac{1}{4} H \otimes H} = \exp \left( \frac{h}{4} H \otimes H \right) \in U_h^{\hat{\otimes} 2}. \end{aligned}$$

for  $n \geq 0$ .

We use the following *universal  $R$ -matrix* of  $U_h$ ,

$$R^{\pm 1} = \sum_{n \geq 0} \alpha_n^{\pm} \otimes \beta_n^{\pm} \in U_h^{\hat{\otimes} 2},$$



Figure 2: Fundamental tangles, where the orientations of the strands are arbitrary

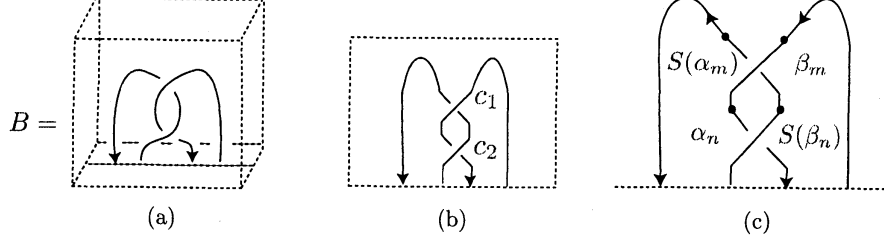


Figure 3: (a) A bottom tangle  $B$ , (b) A diagram  $\tilde{B}$  of  $B$ , (c) The labels associated to a state  $t \in \mathcal{S}(B)$

where we set formally

$$\begin{aligned}\alpha_n \otimes \beta_n (= \alpha_n^+ \otimes \beta_n^+) &= D \left( q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n \right), \\ \alpha_n^- \otimes \beta_n^- &= D^{-1} \left( (-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n \right).\end{aligned}$$

## 2.2 Universal $sl_2$ invariant of bottom tangles

For an  $n$ -component bottom tangle  $T = T_1 \cup \dots \cup T_n$ , we define the universal  $sl_2$  invariant  $J_T \in U_h^{\hat{\otimes} n}$  as follows. We follow the notation in [10].

We choose a diagram  $\tilde{T}$  of  $T$  obtained from the copies of the fundamental tangles depicted in Figure 2, by pasting horizontally and vertically. We denote by  $C(\tilde{T})$  the set of the crossings of  $\tilde{T}$ . For example, for the bottom tangle  $B$  depicted in Figure 3 (a), we can take a diagram  $\tilde{B}$  with  $C(\tilde{B}) = \{c_1, c_2\}$  as depicted in Figure 3 (b). We call a map

$$s: C(\tilde{T}) \rightarrow \{0, 1, 2, \dots\}$$

a *state*. We denote by  $\mathcal{S}(\tilde{T})$  the set of states of the diagram  $\tilde{T}$ .

Given a state  $s \in \mathcal{S}(\tilde{T})$ , we attach labels on the copies of the fundamental tangles in the diagram following the rule described in Figure 4, where “ $S$ ” should be replaced with the identity if the string is oriented downward, and with  $S$  otherwise. For example, for a state  $t \in \mathcal{S}(\tilde{B})$ , we put labels on  $\tilde{B}$  as in Figure 3 (c), where we set  $m = t(c_1)$  and  $n = t(c_2)$ .

We read the labels we have just put on  $\tilde{T}$  and define an element  $J_{\tilde{T},s} \in U_h^{\hat{\otimes} n}$  as follows. Let  $\tilde{T} = \tilde{T}_1 \cup \dots \cup \tilde{T}_n$ , where  $\tilde{T}_i$  corresponds to  $T_i$ . We define the  $i$ th tensorand of  $J_{\tilde{T},s}$  as the product of the labels on  $\tilde{T}_i$ , where the labels are read off along  $T_i$  reversing the orientation, and written from left to right. For example, for the bottom tangle  $B$  and the

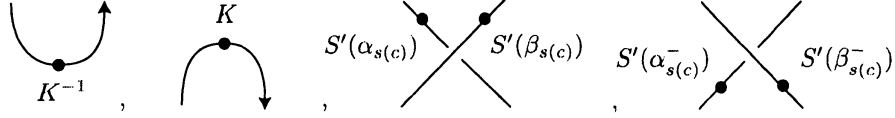


Figure 4: How to place labels on the fundamental tangles

state  $t \in \mathcal{S}(\tilde{B})$  in Figure 3, we have

$$J_{\tilde{B},t} = S(\alpha_m)S(\beta_n) \otimes \alpha_n \beta_m.$$

Here, we identify the labels  $S'(\alpha_i^\pm)$  and  $S'(\beta_i^\pm)$  with the first and the second tensorands, respectively, of the element  $S'(\alpha_i^\pm) \otimes S'(\beta_i^\pm) \in U_h^{\hat{\otimes} 2}$ . Also we identify the label  $K^{\pm 1}$  with the element  $K^{\pm 1} \in U_h$ . Thus  $J_{\tilde{T},s}$  is a well-defined element in  $U_h^{\hat{\otimes} n}$ . For example, we have

$$\begin{aligned} J_{\tilde{B},t} &= S(\alpha_m)S(\beta_n) \otimes \alpha_n \beta_m \\ &= \sum q^{\frac{1}{2}m(m-1)} q^{\frac{1}{2}n(n-1)} S(D'_1 \tilde{F}^{(m)} K^{-m}) S(D'_2 e^n) \otimes D'_2 \tilde{F}^{(n)} K^{-n} D'_1 e^m \\ &= (-1)^{m+n} q^{-n+2mn} D^{-2} (\tilde{F}^{(m)} K^{-2n} e^n \otimes \tilde{F}^{(n)} K^{-2m} e^m) \in U_h^{\hat{\otimes} 2}, \end{aligned}$$

where  $D = \sum D'_1 \otimes D'_1 = \sum D'_2 \otimes D'_2$ . Note that  $J_{\tilde{T},s}$  depends on the choice of the diagram. Set

$$J_T = \sum_{s \in \mathcal{S}(\tilde{T})} J_{\tilde{T},s}.$$

As is well known [7],  $J_T$  does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

### 3 Result for the universal $sl_2$ invariant

In this section, we give the main result for the universal  $sl_2$  invariant of Brunnian bottom tangles. Before that, we recall  $\mathbb{Z}[q, q^{-1}]$ -subalgebras of  $U_h$  and several results for the universal  $sl_2$  invariant of algebraically-split bottom tangles.

#### 3.1 $\mathbb{Z}[q, q^{-1}]$ -subalgebras of $U_h$

We recall  $\mathbb{Z}[q, q^{-1}]$ -subalgebras of  $U_h$ .

Let  $U_{\mathbb{Z},q} \subset U_h$  denote the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by  $K, K^{-1}, \tilde{E}^{(n)}$ , and  $\tilde{F}^{(n)}$  for  $n \geq 1$ , which is a  $\mathbb{Z}[q, q^{-1}]$ -version of Lusztig's integral form (cf. [6, 9]).

Let  $\mathcal{U}_q \subset U_{\mathbb{Z},q}$  denote the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by  $K, K^{-1}, e$ , and  $\tilde{F}^{(n)}$  for  $n \geq 1$ .

Let  $\bar{U}_q \subset \mathcal{U}_q$  denote the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by  $K, K^{-1}, e$  and  $f$ , which is a  $\mathbb{Z}[q, q^{-1}]$ -version of the integral form defined by De Concini and Procesi (cf. [1, 9]).

For  $X = U_{\mathbb{Z},q}, \mathcal{U}_q, \bar{U}_q$ , let  $X^{\text{ev}}$  denote the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra of  $U_h$  defined by the same generators as  $X$  except that  $K^{\pm 2}$  replaces  $K^{\pm 1}$ .

To summarize, we have the following inclusions of the subalgebras of  $U_h$ .

$$\begin{array}{ccccc} \bar{U}_q^{\text{ev}} & \subset & \mathcal{U}_q^{\text{ev}} & \subset & U_{\mathbb{Z},q}^{\text{ev}} \\ \cap & & \cap & & \cap \\ \bar{U}_q & \subset & \mathcal{U}_q & \subset & U_{\mathbb{Z},q} \subset U_h \end{array}$$

We recall the completion  $\tilde{\mathcal{U}}_q^{\text{ev}}$  of  $\mathcal{U}_q^{\text{ev}}$  in  $U_h$  and its completed tensor powers  $(\tilde{\mathcal{U}}_q^{\text{ev}})^{\hat{\otimes} n}$  for  $n \geq 0$ .

For  $p \geq 0$ , let  $\mathcal{F}_p(\mathcal{U}_q^{\text{ev}})$  be the two-sided ideal in  $\mathcal{U}_q^{\text{ev}}$  generated by  $e^p$ . Let  $\tilde{\mathcal{U}}_q^{\text{ev}}$  be the completion of  $\mathcal{U}_q^{\text{ev}}$  in  $U_h$  with respect to the decreasing filtration  $\{\mathcal{F}_p(\mathcal{U}_q^{\text{ev}})\}_{p \geq 0}$ , i.e., we define  $\tilde{\mathcal{U}}_q^{\text{ev}}$  as the image of the homomorphism

$$\varprojlim_{p \geq 0} \mathcal{U}_q^{\text{ev}} / \mathcal{F}_p(\mathcal{U}_q^{\text{ev}}) \rightarrow U_h$$

induced by  $\mathcal{U}_q^{\text{ev}} \subset U_h$ .

For  $n \geq 1$  and  $p \geq 0$ , set

$$\mathcal{F}_p((\mathcal{U}_q^{\text{ev}})^{\otimes n}) = \sum_{i=1}^n (\mathcal{U}_q^{\text{ev}})^{\otimes(i-1)} \otimes \mathcal{F}_p(\mathcal{U}_q^{\text{ev}}) \otimes (\mathcal{U}_q^{\text{ev}})^{\otimes(n-i)}.$$

For  $n \geq 1$ , we define  $(\tilde{\mathcal{U}}_q^{\text{ev}})^{\hat{\otimes} n}$  as the completion of  $(\mathcal{U}_q^{\text{ev}})^{\otimes n}$  in  $U_h^{\hat{\otimes} n}$  with respect to the decreasing filtration  $\{\mathcal{F}_p((\mathcal{U}_q^{\text{ev}})^{\otimes n})\}_{p \geq 0}$ .

For a  $\mathbb{Z}[q, q^{-1}]$ -subalgebra  $A$  of  $(\mathcal{U}_q^{\text{ev}})^{\otimes n}$ , we denote by  $\{A\}$  the *closure* of  $A$  in  $(\tilde{\mathcal{U}}_q^{\text{ev}})^{\hat{\otimes} n}$ , i.e., we set

$$\{A\} = \text{Im} \left( \varprojlim_{p \geq 0} (A / (\mathcal{F}_p((\mathcal{U}_q^{\text{ev}})^{\otimes n}) \cap A) \rightarrow U_h^{\hat{\otimes} n} \right).$$

For  $n = 0$ , we define  $(\tilde{\mathcal{U}}_q^{\text{ev}})^{\hat{\otimes} 0} = \mathbb{Z}[q, q^{-1}]$ .

### 3.2 Universal $sl_2$ invariant of algebraically-split bottom tangles, ribbon bottom tangles and boundary bottom tangles

We recall several results for the value of the universal  $sl_2$  invariant of algebraically-split bottom tangles. In what follows, we assume that bottom tangles are 0-framed.

**Theorem 3.1** ([9, Proposition 4.2, Remark 4.7]). *Let  $T$  be an  $n$ -component algebraically-split bottom tangle. For every diagram  $\tilde{T}$  of  $T$  and every state  $s \in \mathcal{S}(\tilde{T})$ , we have*

$$J_{\tilde{T},s} \in (\mathcal{U}_q^{\text{ev}})^{\otimes n}.$$

More precisely, the proof of [9, Proposition 4.2] implies the following proposition.

**Proposition 3.2.** *Let  $T$  be an  $n$ -component algebraically-split bottom tangle. For any diagram  $\tilde{T}$  and any state  $s \in \mathcal{S}(\tilde{T})$ , we have*

$$J_{\tilde{T},s} \in \mathcal{F}_{|s|}((\mathcal{U}_q^{\text{ev}})^{\otimes n}),$$

where we set  $|s| = \max\{s(c) \mid c \in C(\tilde{T})\}$ .

Theorem 3.1 and Proposition 3.2 imply the following theorem, which was first proved by Habiro [3] in a different way.

**Theorem 3.3** (Habiro [3]). *For an  $n$ -component algebraically-split bottom tangle  $T$ , we have*

$$J_T \in (\tilde{\mathcal{U}}_q^{\text{ev}})^{\tilde{\otimes} n}.$$

In [3], Habiro denoted by  $(\bar{U}_q^{\text{ev}})^{\sim \tilde{\otimes} n}$  the closure  $\{(\bar{U}_q^{\text{ev}})^{\otimes n}\}^\sim$  of  $(\bar{U}_q^{\text{ev}})^{\otimes n}$  in  $(\tilde{\mathcal{U}}_q^{\text{ev}})^{\tilde{\otimes} n}$ . In [9] and [10], we defined a refined completion  $(\bar{U}_q^{\text{ev}})^{\wedge \hat{\otimes} n} \subset (\bar{U}_q^{\text{ev}})^{\sim \tilde{\otimes} n}$ , and proved the following theorem, which is an improvement of Theorem 3.3 in the case of ribbon bottom tangles and boundary bottom tangles.

**Theorem 3.4** ([9, 10]). *Let  $T$  be an  $n$ -component ribbon or boundary bottom tangle. Then we have*

$$J_T \in (\bar{U}_q^{\text{ev}})^{\wedge \hat{\otimes} n}.$$

### 3.3 Results for the universal $sl_2$ invariant of Brunnian bottom tangles

The following theorem is the main result of the paper [11], which is an improvement of Theorems 3.1 and 3.3 in the case of Brunnian bottom tangles.

**Theorem 3.5.** *Let  $T$  be an  $n$ -component algebraically-split Brunnian bottom tangle with  $n \geq 2$ .*

- (i) *For each  $i = 1, \dots, n$ , there is a diagram  $\tilde{T}^{(i)}$  of  $T$  such that*

$$J_{\tilde{T}^{(i)}, s} \in (\bar{U}_q^{\text{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z}, q}^{\text{ev}} \otimes (\bar{U}_q^{\text{ev}})^{\otimes n-i}$$

*for any state  $s \in \mathcal{S}(\tilde{T}^{(i)})$ .*

- (ii) *We have  $J_T \in U_{Br}^{(n)}$ , where we set*

$$U_{Br}^{(n)} = \bigcap_{i=1}^n \left\{ \left( (\bar{U}_q^{\text{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z}, q}^{\text{ev}} \otimes (\bar{U}_q^{\text{ev}})^{\otimes n-i} \right) \cap (\mathcal{U}_q^{\text{ev}})^{\otimes n} \right\}.$$

Note that the condition “algebraically-split” in Theorem 3.5 is not necessary when  $n \geq 3$ .

To compare Theorem 3.5 (ii) with Theorems 3.3 and 3.4 for  $n \geq 2$ , we have the following diagram.

$$\begin{array}{lll} \{n\text{-comp. alg. split bottom tangles}\} & \xrightarrow{J} & (\tilde{\mathcal{U}}_q^{\text{ev}})^{\tilde{\otimes} n} \\ & & \cup \\ \{n\text{-comp. alg. split Brunnian bottom tangles}\} & \xrightarrow{J} & U_{Br}^{(n)} \\ & & \cup \\ \{n\text{-comp. ribbon or boundary bottom tangles}\} & \xrightarrow{J} & (\bar{U}_q^{\text{ev}})^{\wedge \hat{\otimes} n} \end{array}$$

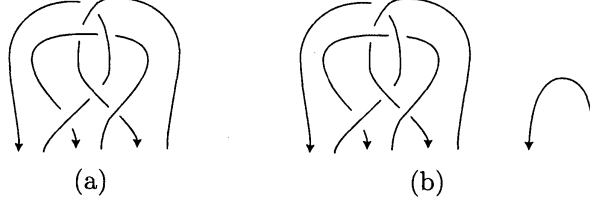


Figure 5: (a) The Borromean bottom tangle  $T_B$ , (b) A bottom tangle  $T'_B$

**Example 3.6.** Let  $T_B$  be the Borromean bottom tangle, which is the Brunnian bottom tangle depicted in Figure 5 (a). Let  $T'_B$  be the bottom tangle as in Figure 5 (b). Note that the bottom tangle  $T'_B$  is not Brunnian but algebraically-split. We have

$$J_{T'_B} = J_{T_B} \otimes 1 \notin \left\{ \left( (\bar{U}_q^{\text{ev}})^{\otimes 3} \otimes U_{\mathbb{Z},q}^{\text{ev}} \right) \cap (\mathcal{U}_q^{\text{ev}})^{\otimes 4} \right\}.$$

## 4 Application to the colored Jones polynomial

In this section, we give an application of Theorem 3.5 to the colored Jones polynomial of Brunnian links (Theorem 4.2). In what follows, we assume that links are 0-framed.

### 4.1 Colored Jones polynomials of algebraically-split links, ribbon links and boundary links

We recall results for the colored Jones polynomials of algebraically-split links.

For  $m \geq 1$ , let  $V_m$  denote the  $m$ -dimensional irreducible representation of  $U_h$ . Let  $\mathcal{R}$  denote the representation ring of  $U_h$  over  $\mathbb{Q}(q^{\frac{1}{2}})$ , i.e.,  $\mathcal{R}$  is the  $\mathbb{Q}(q^{\frac{1}{2}})$ -algebra

$$\mathcal{R} = \text{Span}_{\mathbb{Q}(q^{\frac{1}{2}})} \{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product.

For an  $n$ -component link  $L$ , take a bottom tangle  $T$  whose closure is  $L$ . For  $X_1, \dots, X_n \in \mathcal{R}$ , the colored Jones polynomial  $J_{L;X_1,\dots,X_n}$  of  $L$  with the  $i$ th component  $L_i$  colored by  $X_i$  is given by

$$J_{L;X_1,\dots,X_n} = (\text{tr}_q^{X_1} \otimes \dots \otimes \text{tr}_q^{X_n})(J_T) \in \mathbb{Q}(q^{\frac{1}{2}}).$$

Habiro [3] studied the following element in  $\mathcal{R}$

$$\tilde{P}'_l = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R}, \quad (1)$$

for  $l \geq 0$ .

Recall the notation  $\{l\}_{q,i} = \{l\}_q \{l-1\}_q \dots \{l-i+1\}_q$  for  $l \in \mathbb{Z}$ ,  $i \geq 0$ . Theorem 3.3 implies the following result.



**Theorem 4.1** (Habiro [3]). *Let  $L$  be an  $n$ -component algebraically-split link. For  $l_1, \dots, l_n \geq 0$ , we have*

$$J_{L; \tilde{P}'_{l'_1}, \dots, \tilde{P}'_{l'_n}} \in Z_a^{(l_1, \dots, l_n)}. \quad (2)$$

Here we set

$$Z_a^{(l_1, \dots, l_n)} = \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}],$$

where  $l_{\max} = \max(l_1, \dots, l_n)$ .

For  $l \geq 0$ , let  $I_l$  denote the ideal in  $\mathbb{Z}[q, q^{-1}]$  generated by  $\{l - k\}_q! \{k\}_q!$  for  $k = 0, \dots, l$ . Theorem 3.4 implies the following result.

**Theorem 4.2** ([9, 10]). *Let  $L$  be an  $n$ -component ribbon link or boundary link. For  $l_1, \dots, l_n \geq 0$ , we have*

$$J_{L; \tilde{P}'_{l'_1}, \dots, \tilde{P}'_{l'_n}} \in Z_{r,b}^{(l_1, \dots, l_n)}. \quad (3)$$

Here we set

$$\begin{aligned} Z_{r,b}^{(l_1, \dots, l_n)} &= \left( \prod_{1 \leq i \leq n, i \neq i_M} I_{l_i} \right) \cdot Z_a^{(l_1, \dots, l_n)} \\ &= \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q} \prod_{1 \leq i \leq n, i \neq i_M} I_{l_i}, \end{aligned}$$

where  $l_{\max} = \max(l_1, \dots, l_n)$  and  $i_M$  is an integer such that  $l_{i_M} = l_{\max}$ .

## 4.2 Result for the colored Jones polynomial of Brunnian links

The following theorem is an application of Theorem 3.5 to the colored Jones polynomial of Brunnian links.

**Theorem 4.3.** *Let  $L$  be an  $n$ -component algebraically-split Brunnian link with  $n \geq 2$ . For  $l_1, \dots, l_n \geq 0$ , we have*

$$J_{L; \tilde{P}'_{l'_1}, \dots, \tilde{P}'_{l'_n}} \in Z_{Br}^{(l_1, \dots, l_n)}. \quad (4)$$

Here we set

$$Z_{Br}^{(l_1, \dots, l_n)} = \frac{\{2l_{\max} + 1\}_{q, l_{\max} + 1}}{\{1\}_q \{l_{\min}\}_q!} \prod_{1 \leq i \leq n, i \neq i_M, i_m} I_{l_i},$$

where  $l_{\max} = \max(l_1, \dots, l_n)$ ,  $l_{\min} = \min(l_1, \dots, l_n)$  and  $i_M, i_m$ ,  $i_M \neq i_m$ , are two integers such that  $l_{i_M} = l_{\max}$ ,  $l_{i_m} = l_{\min}$ , respectively.

Note that an algebraically-split Brunnian link satisfies both (2) and (4). In fact, there is no inclusion which satisfies for all  $l_1, \dots, l_n \geq 0$  between  $Z_a^{(l_1, \dots, l_n)}$  and  $Z_{Br}^{(l_1, \dots, l_n)}$ . Thus we have a refinement of Theorem as follows.

**Theorem 4.4.** *Let  $L$  be an  $n$ -component algebraically-split Brunnian link with  $n \geq 2$ . For  $l_1, \dots, l_n \geq 0$ , we have*

$$J_{L; \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}} \in \tilde{Z}_{Br}^{(l_1, \dots, l_n)}.$$

Here we set

$$\tilde{Z}_{Br}^{(l_1, \dots, l_n)} = Z_a^{(l_1, \dots, l_n)} \cap Z_{Br}^{(l_1, \dots, l_n)}.$$

Comparing Theorem 4.4 with Theorems 4.1 and 4.2 for  $n \geq 2$ , we have the following diagram.

$$\begin{array}{ccc} \{n\text{-comp. alg. split links}\} & \xrightarrow{J_{*, \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}}} & Z_a^{(l_1, \dots, l_n)} \\ & & \cup \\ \{n\text{-comp. alg. split Brunnian links}\} & \xrightarrow{J_{*, \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}}} & \tilde{Z}_{Br}^{(l_1, \dots, l_n)} \\ & & \cup \\ \{n\text{-comp. ribbon or boundary links}\} & \xrightarrow{J_{*, \tilde{P}'_{l_1}, \dots, \tilde{P}'_{l_n}}} & Z_{r,b}^{(l_1, \dots, l_n)} \end{array}$$

**Remark 4.5.** In fact, the ideals  $Z_a^{(l_1, \dots, l_n)}$ ,  $Z_{r,b}^{(l_1, \dots, l_n)}$ ,  $Z_{Br}^{(l_1, \dots, l_n)}$  and  $\tilde{Z}_{Br}^{(l_1, \dots, l_n)}$  are principal, each generated by a product of cyclotomic polynomials. See [12] for details and examples.

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Research Institute for Mathematical Sciences  
Kyoto University  
Kyoto 606-8502  
JAPAN  
E-mail address: `sakie@kurims.kyoto-u.ac.jp`

京都大学・数理解析研究所 鈴木 咲衣